Lecture 10 : Chain Rule

(Please review Composing Functions under Algebra/Precalculus Review on the class webpage.)

Here we apply the derivative to composite functions. We get the following rule of differentiation:
The Chain Rule : If g is a differentiable function at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$\boxed{F'(x) = f'(g(x))g'(x).}$$

In Leibniz notation If $y = f(u)$ and $u = g(x)$ are both differentiable functions, then

$$\boxed{\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.}$$

It is not difficult to se why this is true, if we examine the average change in the value of $F(x)$ that results from a small change in the value of x:

$$\frac{F(x+h) - F(x)}{h} = \frac{f(g(x+h)) - f(g(x))}{h} = \frac{f(g(x+h)) - f(g(x))}{g(x+h) - g(x)} \cdot \frac{g(x+h) - g(x)}{h}$$

or if we let $u = g(x)$ and $y = F(x) = f(u)$, then

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x}$$

if $g(x+h) - g(x) = \Delta u \neq 0$. When we take the limit as $h \to 0$ or $\Delta x \to 0$, we get

$$F'(x) = f'(g(x))g'(x)$$

or

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

Example Find the derivative of $F(x) = \sin(2x + 1)$.

Step 1: Write $F(x)$ as $F(x) = f(g(x))$ or $y = F(x) = f(u)$, where $u = g(x)$.

Step 2: working from the outside in, we get
$F'(x) = f'(g(x))g'(x) =$
or using u, we get
$F'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$.

Example Let $g(x) = \sqrt{(x^3 + x^2 + 1)^3}$, Find $h'(x)$.

There is a general pattern with differentiating a power of a function that we can single out as:

The Chain Rule and Power Rule combined: If n is any real number and $u = g(x)$ is differentiable, then

$$\frac{d}{dx}(u^n) = nu^{n-1}\frac{du}{dx}$$

or

$$\frac{d}{dx}((g(x))^n) = n(g(x))^{n-1}g'(x).$$

Example Differentiate the following function:

$$f_1(x) = \sin^{100} x.$$

We can combine the chain rule with the other rules of differentiation:

Example Differentiate $h(x) = (x+1)^2 \sin x$.

Example Find the derivative of the function

$$k(x) = \frac{(x^3+1)^{100}}{x^2+2x+5}.$$

Sometimes we have to use the chain rule more than once. The following can be proven by repeatedly applying the above result on the chain rule :

Expanded Chain Rule If h is differentiable at x, g is differentiable at $h(x)$ and f is at $g(h(x))$, then the composite function $G(x) = f(g(h(x)))$ is differentiable at x and

$$G'(x) = f'(g(h(x)))\ g'(h(x))\ h'(x).$$

Alternatively, letting $v = h(x)$, $u = g(v) = g(h(x))$ and $y = G(x) = f(u)$, we get

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}.$$

Example Let $F(x) = \cos(\sin(x^2 + \pi))$, Find $F'(x)$.

What is the equation of the tangent line to the graph of $f(x)$ at $x = 0$.

More Examples

Example(Old Exam Question Fall 2007) Find the derivative of

$$h(x) = x^2 \cos(\sqrt{x^3 - 1} + 2).$$

Example Find the derivative of

$$F(x) = \frac{1}{\sqrt{x^2 + x + 1}}$$

$$F(x) = \frac{1}{\sqrt{x^2 + x + 1}} = (x^2 + x + 1)^{-1/2}.$$

By the chain rule,

$$F'(x) = \frac{-1}{2}(x^2 + x + 1)^{-3/2}(2x + 1) = \frac{-(2x + 1)}{2(x^2 + x + 1)^{3/2}}.$$

Example Find the derivative of $L(x) = \sqrt{\frac{x-1}{x+2}}$.

Here we use the chain rule followed by the quotient rule. We have

$$L(x) = \sqrt{\frac{x-1}{x+2}} = \left(\frac{x-1}{x+2}\right)^{1/2}.$$

Using the chain rule, we get

$$L'(x) = \frac{1}{2}\left(\frac{x-1}{x+2}\right)^{-1/2} \frac{d}{dx}\left(\frac{x-1}{x+2}\right).$$

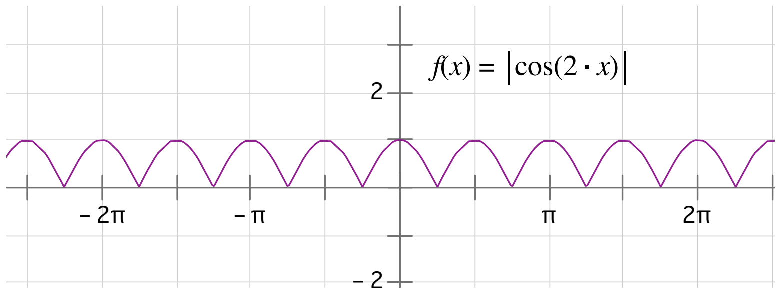
Using the quotient rule for the derivative on the right, we get

$$L'(x) = \frac{1}{2}\left(\frac{x-1}{x+2}\right)^{-1/2} \left[\frac{(x+2) - (x-1)}{(x+2)^2}\right] = \frac{1}{2}\left(\frac{x-1}{x+2}\right)^{-1/2} \left[\frac{3}{(x+2)^2}\right].$$

Example

$$f(x) = \sqrt{\cos^2(2x)}$$

Find $f'(0)$. (Note that this is an interesting function, in fact $f(x) = |\cos(2x)|$ which you can graph by sketching the graph of $\cos(2x)$ and then flipping the negative parts over the x-axis. Note that the graph has many sharp points, but is smooth at $x = 0$.)



Using the chain rule with the chain
$y = f(x) = \sqrt{\cos^2(2x)} = \sqrt{u}$, $\quad u = \cos^2(2x) = (v)^2$, $\quad v = \cos(2x) = \cos(w)$, $\quad w = 2x$, we get

$$f'(x) = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{dx} = \frac{1}{2}u^{-1/2} \cdot 2v \cdot [-\sin(w)] \cdot 2 =$$

$$\frac{1}{2\sqrt{\cos^2(2x)}} \cdot 2\cos(2x) \cdot [-\sin(2x)] \cdot 2 = \frac{-2\cos(2x)\sin(2x)}{\sqrt{\cos^2(2x)}}.$$